

SUFFICIENT CONDITIONS FOR THE PROBLEM OF BOLZA IN THE CALCULUS OF VARIATIONS*

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1. Introduction. Let the end points of the arcs

$$(1:1) \quad y_i = y_i(x) \quad (x^1 \leq x \leq x^2; i = 1, \dots, n)$$

be denoted by the symbols $(x^1, y_1^1, \dots, y_n^1)$ and $(x^2, y_1^2, \dots, y_n^2)$. The problem to be considered is that of finding in a class of arcs (1:1) and sets $(\alpha) = (\alpha_1, \dots, \alpha_r)$ satisfying the differential equations and end conditions

$$(1:2) \quad \phi_\beta(x, y, y') = 0 \quad (\beta = 1, \dots, m < n),$$

$$(1:3) \quad x^s = x^s(\alpha), y_i^s = y_i^s(\alpha) \quad (s = 1, 2)$$

one which minimizes a functional of the form

$$J = \theta(\alpha) + \int_{x^1}^{x^2} f(x, y, y') dx.$$

This problem was first formulated by Bolza (II, p. 431)‡ and will be called the *problem of Bolza*. The formulation here given is due to Morse and Myers (VI, p. 236). Of special importance is the case in which the end conditions (1:3) are of the form

$$(1:4) \quad \begin{aligned} x^1 &= x^1(\alpha_1, \dots, \alpha_\rho), & y_i^1 &= y_i^1(\alpha_1, \dots, \alpha_\rho), \\ x^2 &= x^2(\alpha_{\rho+1}, \dots, \alpha_r), & y_i^2 &= y_i^2(\alpha_{\rho+1}, \dots, \alpha_r) \end{aligned}$$

and the function $\theta(\alpha)$ is of the form

$$\theta(\alpha) = \theta^1(\alpha_1, \dots, \alpha_\rho) - \theta^2(\alpha_{\rho+1}, \dots, \alpha_r).$$

The latter problem will be called the *problem of Bolza with separated end conditions*. In the proof of Theorem 9:2 below it will be shown that the two problems are equivalent not only in the sense that each can be transformed into one of the other type but also in the sense that the theory of the one can be deduced from that of the other.

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‡ Roman numerals in parentheses refer to the list of references at the end of this paper.

Sufficient conditions for a minimum in the problems of Bolza were first given by Morse (VIII) and later by Bliss (IX) and Hu (XIX). However, the normality assumptions, which they make, prevent these conditions from being applicable without further modification to the problem of Mayer (III; XI), to the case in which the functions ϕ_β contain no derivatives, and to a number of other problems. Sufficient conditions for the problem of Mayer have been deduced by Bliss and Hestenes (XVII; XVIII) who make similar restrictive normality assumptions. In §9 below we give for the first time sets of sufficient conditions for the problem of Bolza containing no normality assumptions whatsoever. We merely assume the existence of a set of multipliers of the form $\lambda_0=1, \lambda_\beta(x)$ with which the arc g under consideration satisfies suitable analogues of the usual sufficiency conditions. It is clear that the results of the present paper are applicable at once to the problem of Mayer and thus unify the problems of Bolza and Mayer so that they are equivalent not only in the sense that each can be transformed into one of the other type but also in the sense that the theory for the one can be deduced from that of the other without further modification. The results of this paper also show that the classical problem of Mayer can be considered as a problem of Lagrange with one variable end point (cf. I, p. 224). Moreover by the use of a device given by Bliss (V, p. 703) the results here given can be applied to the case in which the functions ϕ_β contain no derivatives. One obtains thereby an extension of the results given by Bower (XX).

In order to obtain the sufficient conditions here given we derive in §4 a new analogue of the necessary condition of Mayer for the problem of Bolza with separated end conditions. A similar condition has been given by Carrier (XII, p. 699) for parametric problems without differential side conditions and with special end conditions. The methods of Carrier, however, do not seem to be readily extensible to the problem of Bolza without making stringent normality assumptions. A very special case of this necessary condition has been given by Bliss for variable end point problems in the plane (IV, pp. 324–6).

The sufficiency proof given in §§ 6 and 9 below is new and is simpler than those given hitherto for the problems of Bolza. It is a direct extension of the classical method used for fixed end point problems and does not make use of the famous theorem of Hahn (IX, p. 267).

The author has made extensive use of the papers of Bliss and Morse listed at the end of this paper.

2. **First necessary conditions.** Let us suppose that we have given an open region \mathfrak{R} of points (x, y, y') in which the functions f, ϕ_β have continuous derivatives of the first three orders. A set (x, y, y') is said to be *admissible* if it is in \mathfrak{R} and satisfies the equations $\phi_\beta=0$. A *differentiably admissible arc* is

a continuous arc having a continuously turning tangent except possibly at a finite number of points on it and having all of its elements (x, y, y') admissible. A differentially admissible arc (1:1) and a set of constants $(\alpha) = (\alpha_1, \dots, \alpha_r)$ satisfying the end conditions (1:3) are said to form an *admissible arc*.

We center our attention on a particular admissible arc g and propose to find under what conditions g will surely furnish a minimum to J relative to neighboring admissible arcs. We assume that the matrix $\|\phi_{\beta y_i'}\|$ has rank m on g and that the set (α) belonging to g is the set $(\alpha) = (0)$. The functions $\theta(\alpha), x^s(\alpha), y_i^s(\alpha)$ ($s=1, 2$) are assumed to have continuous first and second partial derivatives near $(\alpha) = (0)$.

The tensor analysis summation convention will be used throughout.

The following necessary condition is well known and has been established by Morse and Myers (VI, p. 245) and by Bliss and Schoenberg (X, pp. 681-3) and by others.

THEOREM 2:1. *If g affords a minimum to J then there exist for it constants c_1, \dots, c_n and a function $F = \lambda_0 f + \lambda_\beta(x) \phi_\beta$ ($\beta = 1, \dots, m$) such that the equations*

$$(2:1) \quad F_{y_i'} = \int_{x^1}^x F_{y_i'} dx + c_i, \quad \phi_\beta = 0 \quad (i = 1, \dots, n)$$

hold at every point of g . Moreover on g the equation

$$(2:2) \quad [(F - y_i' F_{y_i'}) dx + F_{y_i'} dy_i]_1^2 + \lambda_0 d\theta = 0^*$$

is an identity in $d\alpha_n$ when the differentials $dx^1, dy_1^1, dx^2, dy_2^2, d\theta$ are expressed in terms of the differentials $d\alpha_n$. The multiplier λ_0 is a constant. The multipliers $\lambda_\beta(x)$ are continuous except possibly at values of x defining corners of g . The elements of the set $\lambda_0, \lambda_\beta(x)$ do not vanish simultaneously at any point on g .

By the order q of anormality of g on an interval $x^1 x^2$ relative to the conditions (2:1) is meant the number q of linearly independent sets of multipliers of the form $\lambda_0 = 0, \lambda_\beta(x)$ with which g satisfies the conditions (2:1) on $x^1 x^2$. The order q of g on $x^1 x^2$ cannot exceed the number m of differential equations $\phi_\beta = 0$. This follows because for every $m+1$ sets of multipliers of the form $\lambda_0 = 0, \lambda_\beta(x)$ there exists at least one linear combination of these sets having constant coefficients not all zero and vanishing at x^1 and hence vanishing for all values of x on $x^1 x^2$. The case $q=0$ on every sub-interval of $x^1 x^2$ has been

* The symbol $[]_1^2$ denotes the value of $[]$ at the final end point 2 on g minus its value at the initial end point 1 on g .

treated by Morse and Bliss. In this case g is said to be normal on every sub-interval.

Carathéodory (XV, XVI) has shown that in the analytic case the order q of anormality of g is the same on every sub-interval of x^1x^2 . In the non-analytic case this is not necessarily true, as will be seen in the example given at the end of §9.

By the order p of anormality of g relative to the conditions (2:1) and (2:2) is meant the number p of linearly independent sets of multipliers of the form $\lambda_0=0, \lambda_\beta(x)$ with which g satisfies the conditions (2:1) and (2:2). Clearly the order p of g cannot exceed the order q of g on the interval x^1x^2 defined by its end points. If $p=0$ then g is said to be normal. In the normal case there exists an infinity of admissible arcs in every neighborhood of g . In the anormal case this is not necessarily true. Moreover for a normal minimizing arc g there exists a unique set of multipliers of the form $\lambda_0=1, \lambda_\beta(x)$ satisfying the conditions of Theorem 2:1 (V, pp. 693-5).

We have the following analogue of the necessary condition of Weierstrass which has been established by Graves (XIII, p. 751).

THEOREM 2:2. *If g is a normal minimizing arc then at each element (x, y, y', λ) on g the inequality*

$$E(x, y, y', \lambda, Y') \geq 0$$

must hold for every admissible set $(x, y, Y') \neq (x, y, y')$ whose matrix $\|\phi_{\beta y'_i}(x, y, Y')\|$ has rank m , where

$$(2:3) \quad E(x, y, y', \lambda, Y') = F(x, y, Y', \lambda) - F(x, y, y', \lambda) - (Y'_i - y'_i)F_{y'_i}(x, y, y', \lambda).$$

The analogue of the necessary condition of Clebsch given in Theorem 4:5 below can also be obtained from Theorem 2:2 by the arguments given by Bliss (V, pp. 718-9).

An *extremal* arc is defined to be a differentially admissible arc and a set of multipliers

$$y_i = y_i(x), \lambda_\beta = \lambda_\beta(x) \quad (x^1 \leq x \leq x^2)$$

having continuous derivatives $y'_i, y''_i, \lambda'_\beta$ and satisfying with $\lambda_0=1$ the *Euler-Lagrange equations*

$$(2:4) \quad (d/dx)F_{y'_i} - F_{y_i} = 0, \phi_\beta = 0.$$

An extremal is said to be *non-singular* if the determinant

$$\begin{vmatrix} F_{y'_i y'_k} & \phi_{\beta y'_i} \\ \phi_{\beta y'_k} & 0 \end{vmatrix}$$

is different from zero at each element (x, y, y', λ) on it. A study of the extremal family has been made by Bliss (V, p. 687).

In the sequel it will be understood that the admissible arc g under consideration is an extremal arc satisfying the conditions (2:1) and (2:2) of Theorem 2:1 unless otherwise expressly stated.

3. The second variation and the accessory minimum problem. In this section we are concerned with the functional

$$J_2(\eta, w) = b_{hl}w_h w_l + \int_{x^1}^{x^2} 2\omega(x, \eta, \eta') dx \quad (h, l = 1, \dots, r)$$

evaluated along the extremal g , where (s not summed; $s = 1, 2$)

$$\begin{aligned} b_{hl} &= \theta_{hl} + [(F_x - y'_i F_{y_i}) x_h^s x_l^s + (F - y'_i F_{y_i'}) x_{hl}^s \\ &\quad + F_{y_i} (x_h^s y'_{il} + x_l^s y'_{ih}) + F_{y_i'} y'_{ihl}]_1^2, \\ 2\omega &= F_{y_i y_k} \eta_i \eta_k + 2F_{y_i y_k'} \eta_i \eta'_k + F_{y_i' y_k'} \eta'_i \eta'_k \quad (i, k = 1, \dots, n). \end{aligned}$$

Here the symbols x^s, y_i^s denote the functions $x^s(\alpha), y_i^s(\alpha)$ and the subscripts h and l denote differentiation with respect to α_h and α_l respectively at $(\alpha) = (0)$. The matrix $\|b_{hl}\|$ is symmetric. The functions $\eta_i(x)$ are assumed to possess continuous derivatives except possibly at a finite number of values of x on the interval $x^1 x^2$ and to satisfy with the constants w_h the equations

$$\begin{aligned} \Phi_\beta(x, \eta, \eta') &= \phi_{\beta y_i} \eta_i + \phi_{\beta y_i'} \eta'_i = 0, \\ \eta_i^s &= c_{ih}^s w_h \quad (s = 1, 2; h = 1, \dots, r) \end{aligned}$$

evaluated along g , where $c_{ih}^s = y_{ih}^s(0) - y'_i(x^s) x_h^s(0)$ (s not summed). Such a set η_i, w_h is called a set of *admissible variations* for g . The functional $J_2(\eta, w)$ is called the *second variation* of the functional J along g (cf. VIII, pp. 520-1).

THEOREM 3:1. *If g is a normal minimizing extremal arc, then along g the second variation J_2 of J must satisfy the condition $J_2(\eta, w) \geq 0$ for every set of admissible variations η_i, w_h having continuous second derivatives except possibly at a finite number of values of x on the interval $x^1 x^2$ defined by the end points of g .*

The theorem follows readily from the derivation of the second variation given by Morse (VIII, pp. 520-1) provided that we show that for every set of admissible variations η_i, w_h having the continuity properties described in the theorem there exists a one-parameter family of admissible arcs

$$y_i = y_i(x, e), \quad \alpha_h = \alpha_h(e) \quad [x^1(\alpha) \leq x \leq x^2(\alpha)]$$

containing g for $e=0$, having η_i, w_h as its variations along g , and having the following continuity properties. The functions $y_i(x, e), \alpha_h(e)$ have continuous

first and second derivatives with respect to e near $e=0$. The derivatives $y_{ize}, y_{izee}, y_{izx}$ exist and are continuous for values (x, e) near those belonging to g except possibly at a finite number of values of x on x^1x^2 . The existence of such a family is readily established by the methods of Bliss (V, p. 695: cf. VI, p. 249) with suitable modifications in order to obtain the necessary derivatives.

Theorem 3:1 leads us to the study of the *accessory minimum problem*, namely, the problem of minimizing the functional $J_2(\eta, w)$ in the class of admissible variations η_i, w_h . This problem is a problem of Bolza of the type described in §1. From Theorem 2:1 we obtain the following equations which a minimizing arc without corners must satisfy:

$$(3:1) \quad (d/dx)\Omega_{\eta_i'} - \Omega_{\eta_i} = 0, \quad \Phi_\beta = 0 \quad (\beta = 1, \dots, m),$$

$$(3:2) \quad \eta_i^s - c_{ih}^s w_h = 0 \quad (i = 1, \dots, n; s = 1, 2),$$

$$(3:3) \quad \zeta_i^2 c_{ih}^2 - \zeta_i^1 c_{ih}^1 + \mu_0 b_{hl} w_l = 0 \quad (h, l = 1, \dots, r),$$

where $\Omega = \mu_0 \omega + \mu_\beta(x)\Phi_\beta$, $\zeta_i = \Omega_{\eta_i'}$. The equations (3:1) are known as the *accessory equations*, the equations (3:2) as the *secondary end conditions*, the equations (3:3) as the *secondary transversality conditions*. The extremals for this problem will be called *secondary extremals*. The secondary end conditions are said to be *regular* in case the $2n \times r$ -dimensional matrix $\|c_{ih}^s\|$ has rank r on g .

If g is non-singular the equations

$$\zeta_i = \Omega_{\eta_i'}(x, \eta, \eta', \mu), \quad \Phi_\beta(x, \eta, \eta') = 0$$

with $\mu_0=1$ can be solved for the variables η_i', μ_β . The accessory equations with $\mu_0=1$ are then found to be equivalent to equations of the form

$$(3:4) \quad d\eta_i/dx = G_i(x, \eta, \zeta), \quad d\zeta_i/dx = H_i(x, \eta, \zeta),$$

where G_i, H_i are linear in the variables η_i, ζ_i (V, p. 727). For every pair of solutions η_i, ζ_i and u_i, v_i of these equations the expression $\zeta_i u_i - \eta_i v_i$ is a constant (V, p. 738). If this constant is zero the solutions are said to be *conjugate solutions*. A set of n mutually conjugate linearly independent solutions is said to form a *conjugate system*.

In the separated end point case the quadratic form $b_{hl} w_h w_l$ is of the form

$$b_{\mu\nu}^1 w_\mu w_\nu - b_{\sigma\tau}^2 w_\sigma w_\tau \quad (\mu, \nu = 1, \dots, \rho; \sigma, \tau = \rho + 1, \dots, r),$$

where

$$(3:5) \quad \begin{aligned} b_{\mu\nu}^1 &= \theta_{\mu\nu}^1 - (F_x - y_i' F_{y_i}) x_\mu^1 x_\nu^1 - (F - y_i' F_{y_i'}) x_\mu^1 \\ &\quad - F_{y_i} (x_\mu^1 y_{i\nu}^1 + x_\nu^1 y_{i\mu}^1) - F_{y_i'} y_{i\mu\nu}^1 \end{aligned}$$

evaluated at the initial point 1 on g and $b_{\sigma\tau}^2$ is a similar expression in $\theta_{\sigma\tau}^2$, x_{σ}^2 , $y_{i\sigma}^2$, $x_{\sigma\tau}^2$, $y_{i\sigma\tau}^2$ evaluated at the final end point 2 on g . The matrices $\|b_{\mu\nu}^1\|$ and $\|b_{\sigma\tau}^2\|$ are symmetric. Moreover the equations (3:2) and (3:3) with $\mu_0 = 1$ can be written in the form

$$(3:6) \quad \eta_i^1 = c_{i\mu}^1 w_{\mu}, \quad \zeta_i^1 c_{i\mu}^1 = b_{\mu\nu}^1 w_{\nu} \quad (\mu, \nu = 1, \dots, \rho),$$

$$(3:7) \quad \eta_i^2 = c_{i\sigma}^2 w_{\sigma}, \quad \zeta_i^2 c_{i\sigma}^2 = b_{\sigma\tau}^2 w_{\tau} \quad (\sigma, \tau = \rho + 1, \dots, r).$$

If the matrix $\|c_{i\mu}^1\|$ has rank ρ then there are n and at most n linearly independent solutions

$$(3:8) \quad \eta_{ik}(x), \zeta_{ik}(x), w_{\mu k} \quad (k = 1, \dots, n)$$

of equations (3:4) and (3:6), as one readily verifies. Moreover the secondary extremals η_{ik} , ζ_{ik} in (3:8) form a conjugate system since at $x = x^1$ we have

$$\begin{aligned} \zeta_{ik} \eta_{ij} - \zeta_{ij} \eta_{ik} &= \zeta_{ik} c_{i\mu}^1 w_{\mu j} - \zeta_{ij} c_{i\mu}^1 w_{\mu k} \\ &= b_{\mu\nu}^1 w_{\mu j} w_{\nu k} - b_{\mu\nu}^1 w_{\mu k} w_{\nu j} = 0 \end{aligned}$$

and since these secondary extremals are linearly independent, as follows readily from the fact that the matrix $\|c_{i\mu}^1\|$ has rank ρ . Similarly if the matrix $\|c_{i\sigma}^2\|$ has rank $r - \rho$, then there are n and at most n linearly independent solutions

$$(3:9) \quad u_{ik}(x), v_{ik}(x), w_{\sigma k} \quad (k = 1, \dots, n)$$

of equations (3:4) and (3:7). It is clear that the secondary extremals u_{ik} , v_{ik} also form a conjugate system.

The following lemma will be useful:

LEMMA 3:1. *The order p of anormality of g is equal to the number of linearly independent secondary extremals η_i , μ_{β} having $(\eta) \equiv (0)$ on $x^1 x^2$ and satisfying the equations (3:3) with the set $(w) = (0)$.*

This result follows because the first n equations (2:4) with $\lambda_0 = 0$ are equivalent to the first n equations (3:1) with $(\eta) \equiv (0)$. Moreover the transversality condition (2:2) with $\lambda_0 = 0$ is equivalent to the conditions (3:3) with $(\eta) \equiv (0)$ and $(w) = (0)$.

Similarly we have

LEMMA 3:2. *The order q of anormality of g on the interval $x^1 x^2$ is equal to the number of linearly independent secondary extremals η_i , μ_{β} having $(\eta) \equiv (0)$ on $x^1 x^2$.*

A further lemma is the following:

LEMMA 3:3. *If u_i, v_i is a secondary extremal having $(u) \equiv (0)$ on x^1x^2 then the relation $v_i\eta_i = \text{constant}$ holds for every differentiably admissible arc $\eta_i(x)$ for the accessory minimum problem.*

Let $\mu_\beta = \lambda_\beta(x)$ be the multipliers belonging to the secondary extremal u_i, v_i . The lemma now follows readily by multiplying the equations of variation

$$\phi_{\beta v_i} \eta_i + \phi_{\beta v_i'} \eta_i' = 0$$

by the functions $\lambda_\beta(x)$, adding, and applying the usual integration by parts with the help of equations (2:4) with $\lambda_0 = 0$.

An important consequence of Lemma 3:3 is that the accessory minimum problem can be modified so that its admissible arcs are all normal. This can be done by replacing the secondary end conditions (3:2) by the conditions

$$(3:10) \quad \eta_i^1 = c_{i\hbar}^1 w_\hbar, \quad \eta_i^2 = c_{i\hbar}^2 w_\hbar + \zeta_{i\gamma}^2 w_{r+\gamma} \quad (\gamma = 1, \dots, p)$$

where p is the order of anormality of g and $\eta_{i\gamma}, \zeta_{i\gamma}$ are p linearly independent secondary extremals having the properties described in Lemma 3:1. We may suppose that these secondary extremals have been chosen so that the columns of the matrix $\|\zeta_{i\gamma}^2\|$ are normed and orthogonalized. By Lemma 3:1 we have

$$(3:11) \quad \zeta_{i\gamma}^2 c_{i\hbar}^2 - \zeta_{i\gamma}^1 c_{i\hbar}^1 = 0 \quad (\gamma = 1, \dots, p).$$

Multiplying the equations (3:10) by the values $-\zeta_{i\gamma}^1, \zeta_{i\gamma}^2$ and adding, it is found with the help of equations (3:11) and Lemma 3:3 that the equations

$$0 = -\zeta_{i\gamma}^1 \eta_i^1 + \zeta_{i\gamma}^2 \eta_i^2 = w_{r+\gamma} \quad (\gamma = 1, \dots, p)$$

hold for every admissible arc $\eta_i, w_\hbar, w_{r+\gamma}$ for the new problem. The new problem is therefore equivalent to the original one. Moreover every admissible arc for the new problem is normal, by Lemma 3:1, since the secondary extremals $\eta_{i\gamma}, \zeta_{i\gamma}$, described above, do not satisfy the analogue of conditions (3:3) with the set $w_\hbar = w_{r+\gamma} = 0$.

LEMMA 3:4. *If g is non-singular then a minimizing arc g_2 for the accessory minimum problem must be an arc defined by a secondary extremal and hence can have no corners.*

As was seen above we may assume that g_2 is normal. Since g_2 is a minimizing arc there exists for it a unique function $\Omega = \omega + \mu_\beta \Phi_\beta$ with which g_2 satisfies the conditions implied by Theorem 2:1. The functions $\zeta_i = \Omega_{\eta_i'}$ are therefore continuous along g_2 . The non-singularity of g and hence of g_2 implies that the functions η_i, μ_β belonging to g_2 define extremal segments between corners of g_2 (V, p. 684). The continuity of the functions ζ_i now implies that

the arc g_2 can have no corners since there is one and only one secondary extremal taking given values η_i^0, ζ_i^0 at a value $x = x^0$ on x^1x^2 . This proves the lemma.

4. Necessary conditions for the second variation to be positive. The second variation $J_2(\eta, w)$ is said to be *positive* along g if the inequality $J_2(\eta, w) \geq 0$ holds for every set of admissible variations η_i, w_k belonging to g . The results of this section will remain valid if we further restrict these variations to have the continuity properties described in Theorem 3:1. The necessary conditions here given must therefore be satisfied if g is to be a *normal* minimizing arc for the original problem.

We have the following necessary condition in the separated end point case. The relations between this condition and those of Currier and Bliss have been explained in §1.

THEOREM 4:1. *If in the separated end point case the extremal g is non-singular, the secondary end conditions are regular, and the second variation J_2 is positive along g , then at each point x^3 on x^1x^2 the inequality*

$$(4:1) \quad (\zeta_{ij}u_{ik} - \eta_{ij}v_{ik})a_jb_k \geq 0 \quad (i, j, k = 1, \dots, n)$$

must hold for every set of constants (a_j, b_k) satisfying the equations

$$(4:2) \quad \eta_{ij}(x^3)a_j = u_{ik}(x^3)b_k$$

where η_{ij}, ζ_{ij} and u_{ik}, v_{ik} are the conjugate systems belonging to the sets (3:8) and (3:9) respectively. The coefficients in the bilinear form (4:1) are constants.

In order to prove the theorem we note that a set of constants (a_j, b_k) satisfying the equations (4:2) determines a broken secondary extremal η_i, ζ_i defined by the equations

$$(4:3) \quad \begin{aligned} \eta_i &= \eta_{ij}a_j, \zeta_i = \zeta_{ij}a_j \text{ on } x^1 \leq x \leq x^3, \\ \eta_i &= u_{ik}b_k, \zeta_i = v_{ik}b_k \text{ on } x^3 \leq x \leq x^2 \end{aligned}$$

and satisfying the conditions (3:6) and (3:7) with the set of constants $w_\mu = w_{\mu j}a_j, w_\sigma = w_{\sigma k}b_k$. Let $\mu_\beta(x)$ be the set of multipliers belonging to the broken extremal η_i, ζ_i . With the help of the formula

$$(4:4) \quad 2\Omega = \eta_i\Omega_{\eta_i} + \eta_i'\Omega_{\eta_i} + \mu_\beta\Omega_{\mu_\beta}$$

and the usual integration by parts it is found that along this broken extremal the second variation J_2 is expressible in the form

$$\begin{aligned} J_2 &= b_{\mu\nu}^1 w_\mu w_\nu - b_{\sigma\tau}^2 w_\sigma w_\tau + \int_{x^1}^{x^3} 2\Omega dx + \int_{x^3}^{x^2} 2\Omega dx \\ &= b_{\mu\nu}^1 w_\mu w_\nu - b_{\sigma\tau}^2 w_\sigma w_\tau + [\eta_i \zeta_i]_1^2 + [\eta_i \zeta_i]_{x^3+0}^{x^2-0}. \end{aligned}$$

By the use of equations (3:6), (3:7), (4:2), and (4:3) it follows readily that

$$\begin{aligned} J_2 &= \eta_i(x^3 + 0)\zeta_i(x^3 - 0) - \zeta_i(x^3 + 0)\eta_i(x^3 - 0) \\ &= (\zeta_{i,j}u_{ik} - \eta_{i,j}v_{ik})a_jb_k. \end{aligned}$$

This formula justifies the inequality (4:1). The last statement in the theorem follows from the remarks made in the paragraph containing the equations (3:4). The theorem is now proved.

Consider now the problem of Bolza in which the end conditions are not necessarily separated. Suppose for the moment that g is non-singular. Let $\eta_{i\rho}, \mu_{\beta\rho}, w_{h\rho}$ ($\rho = 1, \dots, \nu$) be a maximum set of linearly independent secondary extremals and constants (w) satisfying the secondary end conditions (3:2). It is clear that the quadratic form

$$(4:5) \quad Q(z) = J_2(\eta_{i\rho}z_{i\rho}, w_{h\rho}z_{h\rho})$$

in the constants (z_1, \dots, z_ν) must be positive on g if the second variation J_2 is to be positive along g . This proves the first part of the following theorem:

THEOREM 4:2. *If the extremal g is non-singular and the second variation J_2 is positive along g , then the quadratic form (4:5) must be positive on g . Moreover at each point x^3 on x^1x^2 the inequality (4:1) must hold for every set of constants (a_j, b_k) satisfying the equations (4:2), where η_{ij}, ζ_{ij} and u_{ik}, v_{ik} are conjugate systems of secondary extremals having $\eta_{ij}(x^1) = u_{ik}(x^2) = 0$.*

The last part of the theorem is obtained by applying Theorem 4:1 to the case in which the secondary end conditions are of the form $\eta_i^1 = 0, \eta_i^2 = 0$.

A value $x^3 \neq x^1$ is said to define a point 3 conjugate to 1 on g if there exists a secondary extremal $\eta_i = u_i(x), \mu_\beta = \mu_\beta(x)$ having $u_i(x^1) = u_i(x^3) = 0$ but not $(u) \equiv (0)$ on x^1x^3 .

The following necessary condition is a direct extension of a condition given by Bliss (IX, p. 266).

THEOREM 4:3. *If the extremal g is non-singular and the second variation J_2 is positive along g , then the quadratic form (4:5) must be positive on g . Moreover there can be no point 3 conjugate to 1 on g between its end points 1 and 2 defined by a secondary extremal $u_i(x), \rho_\beta(x)$ with $(u') \neq (0)$ at $x = x^3$. If the order q of anormality of g is the same on every sub-interval x^3x^2 of x^1x^2 , then there can be no point 3 conjugate to 1 on g between 1 and 2.*

For if there were a point 3 conjugate to 1 on g between 1 and 2 defined by a secondary extremal u_i, ρ_β , then along the arc

$$\eta_i \equiv u_i(x) \quad (x^1 \leq x \leq x^3), \quad \eta_i \equiv 0 \quad (x^3 \leq x \leq x^2), \quad w_h = 0$$

the second variation would take the value zero (V, p. 726). This arc would therefore be a minimizing arc for the accessory minimum problem and hence could have no corners, by Lemma 3:4. This proves the first statement concerning conjugate points.

In order to prove the last statement of the theorem we note that according to Lemma 3:4 the functions η_i just defined would belong to a secondary extremal η_i, μ_β . The functions η_i would then be identically zero on x^1x^2 since, as one easily sees, Lemma 3:2 and our assumption concerning anormality imply that a secondary extremal η_i, μ_β having $\eta_i \equiv 0$ on x^2x^2 has $\eta_i \equiv 0$ on the whole interval x^1x^2 . It follows that in this case there can be no point 3 on g conjugate to 1 between 1 and 2. This completes the proof of the theorem.

By the *accessory boundary value problem* is meant the equations

$$(4:6) \quad \begin{aligned} (d/dx)\Omega_{\eta_i'} - \Omega_{\eta_i} + \sigma\eta_i &= 0, \quad \Phi_\beta = 0, \\ \eta_i^s &= c_{ih}w_h, \quad \zeta_{ih}^2 - \zeta_{ih}^1 + b_{hi}w_l = 0 \quad (s = 1, 2), \end{aligned}$$

where $\Omega = \omega + \mu_\beta \Phi_\beta$. A set of functions $\eta_i(x), \mu_\beta(x)$ having continuous derivatives $\eta_i', \eta_i'', \eta_\beta'$ and having $(\eta) \neq (0)$ on x^1x^2 is said to form a *characteristic solution* if it satisfies the equations (4:6) with a set of constants w_h, σ . The corresponding value σ is called a *characteristic root*.

We now have the further necessary condition:

THEOREM 4:4. *If the second variation $J_2(\eta, w)$ is positive along the extremal g then there can be no negative characteristic roots of the accessory boundary value problem.*

The proof of this theorem is well known (VIII, p. 524).

We also have the further necessary condition which is an analogue of the necessary condition of Clebsch:

THEOREM 4:5. *If the second variation $J_2(\eta, w)$ is positive along the extremal g , then at each element (x, y, y', λ) on g the inequality*

$$(4:7) \quad F_{\eta_i' \eta_k'} \pi_i \pi_k \geq 0$$

must hold for every set $(\pi) \neq (0)$ which is a solution of the equations $\phi_{\eta_i'} \pi_i = 0$. If g is non-singular then the condition (4:7) holds with the equality sign excluded.

According to the remarks preceding Lemma 3:4 we may suppose that g is normal. The first statement of the theorem can now be obtained by applying Theorem 2:2 to the accessory minimum problem and by the use of Taylor's expansion. The last statement follows readily from well known theorems on quadratic forms.

5. Criteria for conjugate points. A first criterion for conjugate points is the following one:

THEOREM 5:1. *If the extremal g is non-singular and if the functions u_{ij}, v_{ij} ($j = 1, \dots, 2n$) form $2n$ linearly independent secondary extremals for g , then a value $x^3 \neq x^1$ defines a point 3 conjugate to 1 on g if and only if the matrix*

$$(5:1) \quad \left\| \begin{matrix} u_{ij}(x^3) \\ u_{ij}(x^1) \end{matrix} \right\| \quad (i = 1, \dots, n; j = 1, \dots, 2n)$$

has rank less than $2n - q$, where q is the order of anormality of g on the interval x^1x^3 .

The proof of this theorem can be made by the usual methods (V, p. 728) with the help of Corollary 3:2.

THEOREM 5:2. *If the extremal g is non-singular and the order q of anormality of g is the same on every sub-interval x^1x^3 of x^1x^2 , then there exists for g a conjugate system η_{ik}, ζ_{ik} of secondary extremals such that the points 3 conjugate to 1 on g are determined by the zeros $x^3 \neq x^1$ of the determinant $|\eta_{ik}|$.*

If $q = 0$ then it suffices to choose the secondary extremals η_{ik}, ζ_{ik} which take the initial values $\eta_{ik}(x^1) = 0, \zeta_{ik}(x^1) = \delta_{ik}$, where δ_{ik} is the Kronecker delta. This follows readily from Theorem 5:1 by choosing the first n secondary extremals of the set u_{ij}, v_{ij} to be the set η_{ik}, ζ_{ik} .

If $q > 0$ we choose the first n secondary extremals of the set u_{ij}, v_{ij} of Theorem 5:1 such that $u_{ik}(x^1) = 0$ ($k = 1, \dots, n$), $u_{i\gamma}(x) \equiv 0$ ($\gamma = 1, \dots, q$) on x^1x^2 , and such that the columns of the matrix $\|v_{ik}(x^1)\|$ are normed and orthogonalized. The second n secondary extremals of this set are chosen so as to take the initial values $u_{i,n+k}(x^1) = v_{ik}(x^1), v_{i,n+k}(x^1) = 0$. The secondary extremals $\eta_{ik} \equiv u_{i,n+k}, \zeta_{ik} \equiv v_{i,n+k}$ can now be shown to have the properties described in the theorem. An examination of their values at $x = x^1$ will show that they are mutually conjugate. Moreover it is clear that the matrix (5:1) has rank $2n - q$ if and only if the matrix $\|\eta_{i\tau}\| = \|u_{i,n+\tau}\|$ ($\tau = 1, \dots, n - q$) has rank $n - q$ at $x = x^3$. The theorem will now follow from Theorem 5:1 if we show that the determinant $|\eta_{ik}|$ is different from zero if and only if the matrix $\|\eta_{i\tau}\|$ has rank $n - q$. If the determinant $|\eta_{ik}|$ vanishes at a value $x = x^3$ then there exist constants a_k not all zero such that the equations $\eta_{ik}(x^3)a_k = 0$ hold. By the use of Lemma 3:3 and by a consideration of the initial values of the secondary extremals under consideration it is found that

$$0 = v_{i\gamma}(x^3)\eta_{ik}(x^3)a_k = v_{i\gamma}(x^1)\eta_{ik}(x^1)a_k = a_{n-q+\gamma} \quad (\gamma = 1, \dots, q).$$

The matrix $\|\eta_{i\tau}\|$ ($\tau = 1, \dots, n - q$) must therefore have rank less than $n - q$

whenever the determinant $|\eta_{ik}|$ vanishes. The converse is immediate, and the theorem is established.

6. A fundamental sufficiency theorem. The notion of a Mayer field \mathfrak{F} used here is that given by Bliss (V, p. 730). The slope functions and the multipliers belonging to \mathfrak{F} will be denoted by the symbols $p_i(x, y)$, $\lambda_\beta(x, y)$. The Hilbert integral

$$I^* = \int \{F(x, y, p, \lambda)dx + (dy_i - p_i dx)F_{y_i'}(x, y, p, \lambda)\}$$

formed for these functions and $\lambda_0=1$ is independent of the path in \mathfrak{F} . The value of the integral I^* along an extremal of the field is equal to that of the integral

$$(6:1) \quad I = \int f(x, y, y')dx.$$

The Weierstrass E -function $E(x, y, p, \lambda, y')$ is the expression (2:3).

If g is an extremal of a Mayer field then the transversality condition (2:2) for g implies that the equation

$$(6:2) \quad [dI^*]_1^2 + d\theta = 0$$

is an identity in $d\alpha_h$ on g when the differentials $dx^1, dy_1^1, dx^2, dy_1^2, d\theta$ are expressed in terms of the differentials $d\alpha_h$. It follows readily that on g the second differential

$$(6:3) \quad [d^2I^*]_1^2 + d^2\theta$$

is a quadratic form in the variables $d\alpha_h$. With this in mind we can prove the following theorem:

THEOREM 6:1. *Let \mathfrak{F} be a Mayer field in which the inequality*

$$(6:4) \quad E[x, y, p(x, y), \lambda(x, y), y'] > 0$$

holds for every admissible set $(x, y, y') \neq (x, y, p)$. If g is an extremal of the field such that the equation (6:2) is an identity in $d\alpha_h$ on g and such that the quadratic form (6:3) is positive definite on g , then g affords a proper minimum to J relative to admissible arcs C in \mathfrak{F} with sets (α) near (0) .

Let A^1, A^2 be the arcs in \mathfrak{F} defined by the equations

$$(A^s) \quad x^s = x^s(t\alpha), \quad y_i^s = y_i^s(t\alpha) \quad (0 \leq t \leq 1; s = 1, 2)$$

for a set (α) near (0) , where the functions on the right are those appearing in equations (1:3). The condition (6:2) and the positive definiteness of the

quadratic form (6:3) tell us that the set $(\alpha) = (0)$ furnishes a proper minimum to the function

$$W(\alpha) = \theta(\alpha) - \theta(0) + I^*(A^2) - I^*(A^1)$$

relative to sets (α) near (0) .

Suppose now that the set (α) belongs to an admissible arc C in \mathfrak{F} . With the help of the formula (2:3) and the invariant property of the integral I^* it is found that

$$\begin{aligned} I(C) - I(g) &= \int_c E dx + I^*(C) - I^*(g) \\ &= \int_c E dx + I^*(A^2) - I^*(A^1), \end{aligned}$$

where I is the integral (6:1). When the expression $\theta(\alpha) - \theta(0)$ is added to both sides of the last equation, the formula

$$J(C) - J(g) = \int_c E dx + W(\alpha)$$

is obtained. Hence we have $J(C) \geq J(g)$ provided that the set (α) belonging to the arc C is near (0) . The equality holds only in case $(\alpha) = (0)$ and the integral of the E -function vanishes, that is, only in case the ends of C coincide with those of g and the equations $y'_i - p_i = 0$ hold along C . The arc C would then be an extremal of the field and would coincide with g since there is but one extremal of the field through each point of \mathfrak{F} (cf. V, pp. 731-2).

In the sequel we shall apply Theorem 6:1 only to the problem of Bolza with separated end points. If the end conditions are not of the form (1:4) then it is not always possible to construct a field such that the quadratic form (6:3) is positive definite on g . This can be seen by considering the special problem in (xy_1y_2) -space for which $\theta = \alpha$, $f \equiv 0$, $x^1(\alpha) = 0$, $y_1^1(\alpha) = 0$, $y_2^1(\alpha) = -\alpha$, $x^2(\alpha) = 1$, $y_1^2(\alpha) = 0$, $y_2^2(\alpha) = \alpha$, and $\phi_1 = (1 + y_1'^2)^{1/2} - y_2' = 0$. The sufficient conditions given in §9 below, however, are applicable to this problem.

7. **Three lemmas.** Consider first the problem of Bolza with separated end conditions. Suppose that g is non-singular and that the secondary end conditions are regular on g . The arc g will be said to satisfy the *condition IV'* if at each point 3 on g the inequality (4:1) holds subject to the conditions (4:2) and if furthermore the matrix

$$(7:1) \quad \|\zeta_{ij}u_{ik} - \eta_{ij}v_{ik}\|$$

of the coefficients in the bilinear form (4:1) has rank $n - p$ on g , where p is the order of anormality of g . The matrix (7:1) has rank $n - p$ on g if and

only if the equations (3:4), (3:6), (3:7) have no solution (η, ζ, w) other than those described in Lemma 3:1. In the fixed end point case the matrix (7:1) has rank $n - p$ if and only if the end points of g are not conjugate to each other, as is readily verified.

LEMMA 7:1. *If in the separated end point case the extremal g is non-singular and satisfies the condition IV' and if the secondary end conditions are regular on g , then there exists for g a conjugate system U_{ik}, V_{ik} of secondary extremals whose determinant $|U_{ik}(x)|$ is different from zero on the interval $x^1 \leq x \leq x^2$ determined by the end points 1 and 2 of g . Moreover the inequalities*

$$(7:2) \quad \begin{aligned} b_{\mu\nu}^1 z_\mu z_\nu - U_{ik}(x^1) V_{ij}(x^1) a_j a_k &> 0 & (\mu, \nu = 1, \dots, \rho; i, j, k = 1, \dots, n), \\ - b_{\sigma\tau}^2 z_\sigma z_\tau + U_{ik}(x^2) V_{ij}(x^2) b_j b_k &> 0 & (\sigma, \tau = \rho + 1, \dots, r) \end{aligned}$$

hold for every set of constants $(a_k, z_\mu) \neq (0, 0)$ and $(b_k, z_\sigma) \neq (0, 0)$ satisfying the equations

$$(7:3) \quad U_{ik}(x^1) a_k = c_{i\mu}^1 z_\mu, \quad U_{ik}(x^2) b_k = c_{i\sigma}^2 z_\sigma.$$

For, if p is the order of anormality of g , then by virtue of Lemma 3:1 we can select the first p solutions of the sets (3:8) and (3:9) so that on $x^1 x^2$ we have

$$(7:4) \quad \eta_{i\gamma}(x) \equiv u_{i\gamma}(x) \equiv 0, \quad \zeta_{i\gamma}(x) \equiv v_{i\gamma}(x) \quad (\gamma = 1, \dots, p)$$

and so that the columns of the matrix $\|\zeta_{i\gamma}(x^1)\|$ are normed and orthogonalized. We then select the remaining solutions of these sets so that the relations

$$(7:5) \quad \zeta_{i\gamma}(x^1) \zeta_{i\alpha}(x^1) = 0, \quad \zeta_{i\gamma}(x^1) v_{i\alpha}(x^1) = v_{i\gamma}(x^1) v_{i\alpha}(x^1) = 0,$$

$$(7:6) \quad \zeta_{i\alpha} u_{i\beta} - \eta_{i\alpha} v_{i\beta} = \delta_{\alpha\beta} \quad (\alpha, \beta = p + 1, \dots, n)$$

hold, where $\delta_{\alpha\beta}$ is the Kronecker delta. In order to obtain the relations (7:6) we note that, since the conjugate systems η_{ik}, ζ_{ik} and u_{ik}, v_{ik} have the secondary extremals (7:4) in common, it follows that

$$(7:7) \quad \zeta_{i\gamma} u_{ik} - \eta_{i\gamma} v_{ik} = 0, \quad \zeta_{ik} u_{i\gamma} - \eta_{ik} v_{i\gamma} = 0 \quad (\gamma = 1, \dots, p).$$

The determinant

$$(7:8) \quad \left| \zeta_{i\alpha} u_{i\beta} - \eta_{i\alpha} v_{i\beta} \right| \quad (\alpha, \beta = p + 1, \dots, n)$$

must therefore be different from zero if the matrix (7:1) is to have rank $n - p$. The relations (7:6) are now obtained by replacing the solutions $\eta_{i\alpha}, \zeta_{i\alpha}, w_{\mu\alpha}$ by the solutions $\eta_{i\alpha} A_{\alpha\beta}, \zeta_{i\alpha} A_{\alpha\beta}, w_{\mu\alpha} A_{\alpha\beta}$, where the matrix $\|A_{\alpha\beta}\|$ is the reciprocal of the matrix (7:8).

The secondary extremals U_{ik}, V_{ik} taking the initial values

$$(7:9) \quad \begin{aligned} U_{i\gamma}(x^1) &= \zeta_{i\gamma}(x^1), & U_{i\alpha}(x^1) &= \eta_{i\alpha}(x^1) + u_{i\alpha}(x^1) \quad (\gamma = 1, \dots, p), \\ V_{i\gamma}(x^1) &= 0, & V_{i\alpha}(x^1) &= \zeta_{i\alpha}(x^1) + v_{i\alpha}(x^1) \quad (\alpha = p + 1, \dots, n) \end{aligned}$$

can be shown to have the properties described in the theorem. In the first place these secondary extremals are mutually conjugate, as is easily seen, with the help of equations (7:5), (7:6), (7:9) and the conjugacy of the systems η_{ik}, ζ_{ik} and u_{ik}, v_{ik} . Moreover the determinant $|U_{ik}(x)|$ is different from zero on x^1x^2 . In order to prove this we use the relations

$$(7:10) \quad \zeta_{i\gamma}U_{i\epsilon} = \delta_{\gamma\epsilon}, \quad \zeta_{i\gamma}U_{i\alpha} = 0 \quad (\gamma, \epsilon = 1, \dots, p; \alpha = p + 1, \dots, n)$$

which hold identically on x^1x^2 by virtue of Lemma 3:3 and the equations (7:4), (7:7), (7:9) together with the fact that the columns of the matrix $\|\zeta_{i\gamma}(x^1)\|$ are normed and orthogonalized. If now the determinant $|U_{ik}(x)|$ were zero at a point x^3 on x^1x^2 then there would exist constants c_k not all zero such that $U_{ik}(x^3)c_k = 0$. By multiplying these equations by $\zeta_{i\gamma}(x^3)$, adding, and using equations (7:10) it would follow that the constants c_1, \dots, c_p would all be zero and hence that

$$U_{i\alpha}(x^3)c_\alpha = \eta_{i\alpha}(x^3)c_\alpha + u_{i\alpha}(x^3)c_\alpha = 0 \quad (\alpha = p + 1, \dots, n).$$

The equations (4:2) would then be satisfied by the set $a_k = c_k, b_k = -c_k$ and for these constants the bilinear form (4:1) would take the value $a_\alpha b_\alpha = -c_\alpha c_\alpha < 0$ by virtue of the equations (7:6) and (7:7). But this would contradict the condition IV'. The determinant $|U_{ik}|$ must therefore be different from zero on x^1x^2 .

We shall now establish the first of the inequalities (7:2). In order to do this we first note that the constants a_1, \dots, a_p in equations (7:3) are all zero, as can be easily seen, by multiplying the first n of these equations by $\zeta_{i\gamma}(x^1)$, adding, and applying the equations (7:10) and the analogue of equations (3:11) for the separated end point case. We use the abbreviations

$$(7:11) \quad \begin{aligned} \eta_i &= \eta_{i\alpha}a_\alpha, & \zeta_i &= \zeta_{i\alpha}a_\alpha, & w_\mu &= w_{\mu\alpha}a_\alpha, \\ u_i &= u_{i\alpha}a_\alpha, & v_i &= v_{i\alpha}a_\alpha, & z_\mu &= w_\mu + w'_\mu \quad (\alpha = p + 1, \dots, n) \end{aligned}$$

and find that the set η_i, ζ_i, w_μ satisfies the equations (3:6). Moreover by the use of equations (3:6) and (7:3) it follows readily that at $x = x^1$

$$\begin{aligned} U_{ik}a_k - c_{i\mu}z_\mu &= u_i - c_{i\mu}w'_\mu = 0 \\ b_{\mu\nu}w_\nu w'_\mu &= \zeta_i c_{i\mu} w'_\mu = \zeta_i u_i, \\ b_{\mu\nu}w_\nu w_\mu &= \zeta_i c_{i\mu} w_\mu = \zeta_i \eta_i. \end{aligned}$$

With the help of the last two formulas and the equations (7:11) it is found that the first member of the relations (7:2) is expressible in the form

$$\begin{aligned}
 b_{\mu\nu}^1(w_\mu + w'_\mu)(w_\nu + w'_\nu) - (\eta_i + u_i)(\zeta_i + v_i) \\
 = [b_{\mu\nu}^1 w'_\mu w'_\nu - u_i v_i] + [\zeta_i u_i - \eta_i v_i].
 \end{aligned}$$

The second bracket is equal to the sum $a_\alpha a_\alpha$ by virtue of equations (7:6) and (7:11) and is positive unless the constants a_α are all zero, in which case the constants z_μ in equations (7:3) are also all zero since the secondary end conditions are regular. The first bracket in the last equation is positive or zero. For, as a consequence of the regularity of the secondary end conditions there exists for every set of constants w'_μ a secondary extremal η_{i0}, ζ_{i0} satisfying the conditions (3:6) with $w_\mu = w'_\mu$. The set $\eta_{i0}, \zeta_{i0}, w'_\mu$ is expressible linearly with constants c_k in terms of the set (3:8). Moreover it is clear that $\eta_{i0}(x^1) = u_i(x^1)$. Hence we have at $x = x^1$

$$\begin{aligned}
 b_{\mu\nu}^1 w'_\mu w'_\nu - u_i v_i &= \zeta_{i0} \eta_{i0} - u_i v_i \\
 &= \zeta_{i0} u_i - \eta_{i0} v_i = (\zeta_{ij} u_{ik} - \eta_{ij} v_{ik}) c_j a_k
 \end{aligned}$$

and this expression must be positive or zero by IV'. This proves the first inequality (7:2). The second can be established by the same method. The proof of Lemma 7:1 is now complete

We also have the further useful lemma:

LEMMA 7:2. *If the extremal g is non-singular and its end points are not conjugate to each other, then the end points of every differentially admissible arc g_2 for the accessory minimum problem can be joined by a secondary extremal.*

To prove this let q be the order of anormality of g on $x^1 x^2$ and suppose that the first q secondary extremals $u_{i\gamma}, v_{i\gamma}$ ($\gamma = 1, \dots, q$) of the set u_{ij}, v_{ij} ($j = 1, \dots, 2n$) appearing in Theorem 5:1 have been chosen so that $u_{i\gamma} \equiv 0$ on $x^1 x^2$. Since the end points of g are not conjugate, the end values of the remaining $2n - q$ secondary extremals of this set form a set of $2n - q$ linearly independent solutions of the equations

$$v_{i\gamma}(x^2) \eta_i^2 = v_{i\gamma}(x^1) \eta_i^1 \quad (\gamma = 1, \dots, q),$$

by Lemma 3:3, and every solution η_i^1, η_i^2 of these equations is expressible linearly in terms of these $2n - q$ solutions. The end points of g_2 satisfy these equations, by Lemma 3:3. This proves Lemma 7:2.

By the *Clebsch condition III'* is meant the conditions of Theorem 4:5 with the equality sign excluded. The condition III' for g implies that g is non-singular (V, p. 735). We can now prove the following lemma:

LEMMA 7:3. *If the extremal g satisfies the condition III' and if there exists for g a conjugate system U_{ik}, V_{ik} of secondary extremals whose determinant $|U_{ik}(x)|$ is different from zero on x^1x^2 , then every secondary extremal u_i, v_i is an extremal of a Mayer field defined over the region \mathfrak{F}_Ω of points (x, η) whose x -projections lie on the interval x^1x^2 . Moreover the analogue of the condition (6:4) holds in \mathfrak{F}_Ω .*

For, the n -parameter family of secondary extremals

$$(7:12) \quad \eta_i = u_i + U_{ik}a_k, \quad \zeta_i = v_i + V_{ik}a_k$$

contains the extremal u_i, v_i for values $(a) = (0)$ and simply covers the region \mathfrak{F}_Ω . Moreover the Hilbert integral I^* formed for the function 2Ω is independent of the path on the hyperplane $x = x^2$. The family (7:12) therefore defines a field over \mathfrak{F}_Ω (V, p. 733; VII, p. 571). The last statement in the lemma follows at once from the condition III' by the use of Taylor's expansion.

8. Necessary and sufficient conditions for the second variation to be positive definite. The second variation J_2 is said to be *positive definite* along the extremal g if the inequality $J_2(\eta, w) > 0$ is true for every set of admissible variations $(\eta, w) \neq (0, 0)$ belonging to g .

THEOREM 8:1. *If in the separated end point case the extremal g is non-singular and the secondary end conditions are regular on g , then the second variation $J_2(\eta, w)$ is positive definite along g if and only if the conditions III' and IV' hold along g .*

The necessity of the conditions III' and IV' follows at once from Theorems 4:1, 4:5 and the remarks preceding Lemma 7:1. The sufficiency of these conditions follows readily from Lemmas 7:1, 7:3 and Theorem 6:1 applied to the secondary extremal $u_i \equiv v_i \equiv 0$, the conditions (7:2) and (7:3) implying the positive definiteness of the analogue of the quadratic form (6:3).

We now turn to the case in which the end conditions are of the form (1:3). By the *condition V'* is meant the necessary conditions of Theorem 4:2 with the added assumption that the equation $Q(z) = 0$ holds only in case $\eta_{i\rho}z_\rho \equiv 0, w_{h\rho}z_\rho = 0$ on x^1x^2 . The condition V' for g prevents its end points from being conjugate to each other. For if the end points 1 and 2 of g were conjugate then there would exist a secondary extremal η_i, μ_β with $\eta_i(x^1) = \eta_i(x^2) = 0$ and $(\eta) \neq (0)$ on x^1x^2 . The set $\eta_i, \mu_\beta, w_h = 0$ would then be expressible linearly with constants z_ρ in terms of the set $\eta_{i\rho}, \mu_{\beta\rho}, w_{h\rho}$ appearing in the definition of the quadratic form (4:5). For these values of (z) we would have $Q(z) = 0$, as is easily seen, with the help of the formula (4:4) and the usual integration by

parts. It follows that the end points of g cannot be conjugate if the condition V' is to hold along g .

THEOREM 8:2. *If the extremal g is non-singular then the second variation $J_2(\eta, w)$ is positive definite along g if and only if the conditions III' and V' hold along g .*

It is clear that the conditions III', V' are necessary. In order to show that they are sufficient we note first that the condition V' for g implies the condition IV' for the fixed end point case. Lemma 7:1 now tells us that there exists a conjugate system U_{ik}, V_{ik} of secondary extremals whose determinant $|U_{ik}(x)|$ is different from zero on x^1x^2 . From Lemma 7:3 and Theorem 6:1 we conclude that every secondary extremal u_i, v_i affords a proper minimum to the integral

$$I_2 = \int_{x^1}^{x^2} 2\omega(x, \eta, \eta') dx$$

relative to differentially admissible arcs $\eta_i(x)$ joining its end points.

Suppose now that η_j, w_h is an admissible arc for the accessory minimum problem. By Lemma 7:2 there exists a secondary extremal u_i, v_i joining its end points. We have accordingly

$$(8:1) \quad J_2(\eta, w) - J_2(u, w) = I_2(\eta) - I_2(u) \geq 0,$$

the equality being valid only in case $(\eta) \equiv (u)$. From the definition of the quadratic form $Q(z)$ it is clear that there exist constants z_p such that $Q(z) = J_2(u, w)$. From the condition V' and the relation (8:1) we now conclude that $J_2(\eta, w) > 0$ unless $(\eta, w) \equiv (0, 0)$, as was to be proved.

The extremal g will be said to satisfy the *condition VI'* if the quadratic form (4:5) is positive on g and vanishes only in case $\eta_{i\rho}z_p \equiv 0, w_{h\rho}z_p \equiv 0$ on x^1x^2 , and if furthermore there is no point 3 conjugate to the initial point 1 on g . We can now prove the following theorem:

THEOREM 8:3. *If the extremal g is non-singular and the order q of anormality of g is the same on every sub-interval x^1x^3 of x^1x^2 , then the second variation $J_2(\eta, w)$ is positive definite along g if and only if the conditions III', VI' hold along g .*

The proof of this theorem is like that of Theorem 8:2 provided that we can show that there exists for g a conjugate system U_{ik}, V_{ik} of secondary extremals having its determinant $|U_{ik}(x)|$ different from zero on x^1x^2 . This latter result will be obtained by a method first used by Morse (VII, pp. 574-6) for the problem of Lagrange and later adapted to the problem of

Mayer by Bliss and Hestenes (XVII, pp. 320–2). In the proof we suppose that the conjugate system η_{ik}, ζ_{ik} of Theorem 5:2 has been chosen to take the values δ_{ik}, B_{ik} at $x = x^2$, where δ_{ik} is the Kronecker delta and $B_{ik} = B_{ki}$. “Lemma 8:2” of Bliss and Hestenes now holds as before. Similarly “Lemma 8:3” is true, as is easily seen with the help of the following remarks. Although a secondary extremal η_i, ζ_i joining the points $(x, \eta) = (x^1, 0)$ and $(x, \eta) = (x^2, a)$ is not necessarily an extremal of the field it has associated with it a secondary extremal $\eta_i - c_\gamma u_{i\gamma}, \zeta_i - c_\gamma v_{i\gamma}$ ($\gamma = 1, \dots, q$) belonging to the field, where q is the order of anormality of g on x^1x^2 and $u_{i\gamma}, v_{i\gamma}$ are q linearly independent secondary extremals having $u_{i\gamma} \equiv 0$ on x^1x^2 . Moreover the values of the integral “ I_2 ” along these two extremals are the same. The remainder of the proof is now like that of “Theorem 8:1” of Bliss and Hestenes.

We now turn to the accessory boundary value problem. Its characteristic roots are all real (XIV, p. 774; XIX, p. 394). We have the following theorem:

THEOREM 8:4. *If the extremal g is non-singular and the secondary end conditions are regular on g , then the second variation $J_2(\eta, w)$ is positive definite (positive) along g if and only if the condition III' holds along g and the characteristic roots of the accessory boundary value problem are all positive (non-negative).*

According to the remarks preceding Lemma 3:4 we may suppose that g is normal. The theorem then follows from a result given by Hu (XIX, p. 413).

The theorem can also be established with the help of the condition V'. A method will be outlined briefly as follows. We first replace the integrand 2ω in the functional $J_2(\eta, w)$ by $2\omega - \sigma\eta_i\eta_i$ and obtain a functional $J_2(\eta, w, \sigma)$. The preceding theorems concerning the functional $J_2(\eta, w)$ are valid also for the functional $J_2(\eta, w, \sigma)$ when the obvious changes due to the introduction of the parameter σ are made. By an argument like that given by Morse (VIII, pp. 533–4) it is found that for σ sufficiently large and negative the functional $J_2(\eta, w, \sigma)$ will be positive definite relative to sets of admissible variations $(\eta, w) \neq (0, 0)$. Let σ_0 be the least upper bound of the values of σ for which $J_2(\eta, w, \sigma)$ is positive definite. It will be shown below that σ_0 must be finite. We shall now show that σ_0 is a characteristic root. The functional $J_2(\eta, w, \sigma_0)$ must be positive since otherwise there would exist an admissible arc η_i, w_k such that $J_2(\eta, w, \sigma) < 0$ for $\sigma = \sigma_0$ and hence for $\sigma < \sigma_0$ and sufficiently near to σ_0 , which is not the case. If the functional $J_2(\eta, w, \sigma_0)$ were positive definite then by Theorem 8:2 the condition V' would hold for this functional. By the use of Lemma 3:2 applied to sub-intervals of the form x^1x^3 and x^3x^2 and with the help of well known theorems on quadratic forms it could then be shown that the condition V' would hold for the functional $J_2(\eta, w, \sigma)$ for values of σ slightly larger than σ_0 . The functional $J_2(\eta, w, \sigma)$ would then be positive definite for these values of σ , by Theorem 8:2, and σ_0

could not be the least upper bound for such values of σ . It follows that there exists at least one admissible arc η_i, w_h with $(\eta) \neq (0)$ on x^1x^2 such that $J_2(\eta, w, \sigma_0) = 0$. As in the proof of Lemma 3:4 it is seen that this arc has associated with it a set of multipliers $\mu_0 = 1, \mu_\beta(x)$ such that the functions η_i, μ_β define a secondary extremal for the problem of minimizing the functional $J_2(\eta, w, \sigma_0)$ in the class of admissible variations η_i, w_h belonging to g . The functions η_i, μ_β therefore form a characteristic solution and σ_0 a characteristic root.

In order to show that σ_0 is finite we note that there exists at least one set of admissible variations η_i, w_h having $(\eta) \neq (0)$ on x^1x^2 since the accessory minimum problem can be made normal. For this set the functional $J_2(\eta, w, \sigma)$ can be made negative by taking σ sufficiently large and positive. Consequently σ_0 must be finite. This proves the theorem.

9. Sufficient conditions for relative minima. The end conditions (1:3) are said to be *regular* on the admissible arc g under consideration if the matrix of the derivatives of the functions $x^s(\alpha), y_i^s(\alpha)$ has rank r for $(\alpha) = (0)$. The arc g is said to satisfy the *non-tangency condition* if the manifold $y_i^s = y_i^s(x^s)$ ($s = 1, 2$) and the terminal manifold $x^s = x^s(\alpha), y_i^s = y_i^s(\alpha)$ possess no common tangent line at the point $(\alpha) = (0)$ on the terminal manifold. The end conditions are regular and the non-tangency condition holds on g if and only if the secondary end conditions (3:2) are regular on g (VIII, pp. 525-6). No generality is lost in assuming that the end conditions are regular and the non-tangency condition holds on g , as can be seen from the proof of Theorem 9:2 below.

The symbol I will be used to denote the necessary condition of Theorem 2:1. An admissible arc g with a set of multipliers $\lambda_0, \lambda_\beta(x)$ is said to satisfy the *Weierstrass condition* $II'\mathfrak{R}$ if at each element (x, y, y', λ) in a neighborhood \mathfrak{R} of those on g the inequality

$$E[x, y, y', \lambda, Y'] > 0$$

holds for every admissible set $(x, y, Y') \neq (x, y, y')$. The Clebsch condition III' and the conditions IV', V', VI' have been described in §§7 and 8. The last three conditions can readily be expressed in terms of the extremal family in a manner analogous to that given by Bliss (IX, pp. 265-6; cf. XVIII, p. 483).

THEOREM 9:1. *Let g be an admissible arc for the problem of Bolza with separated end conditions. Suppose that the end conditions (1:4) are regular and that the non-tangency condition holds on g . If g has no corners and satisfies the conditions I, II'\mathfrak{R}', III', IV' with a set of multipliers $\lambda_0 = 1, \lambda_\beta(x)$, then g affords a proper strong relative minimum to the functional J .*

From the conditions I and III' we conclude that g is a non-singular extremal since it has no corners (V, p. 735). The theorem will now be established by showing that the hypotheses of Theorem 6:1 are fulfilled.

As a first step we note that g is a member for values $x^1 \leq x \leq x^2$, $a_i = a_{i0}$ ($i = 1, \dots, n$) of an n -parameter family of extremals whose equations in the canonical variables $x, y_i, z_i = F_{y_i'}$, are of the form

$$(9:1) \quad y_i = y_i(x, a_1, \dots, a_n), \quad z_i = z_i(x, a_1, \dots, a_n).$$

The functions $y_i, y_{i,x}, z_i, z_{i,x}$ have continuous first and second derivatives for all values (x, a) in a neighborhood of those belonging to g . The parameters (a) can be chosen so that along g

$$(9:2) \quad y_{ia_k}(x, a_0) \equiv U_{ik}(x), \quad z_{ia_k}(x, a_0) \equiv V_{ik}(x),$$

where the functions U_{ik}, V_{ik} are secondary extremals having the properties described in Lemma 7:1. Moreover the family of extremals (6:1) defines a Mayer field over a neighborhood \mathfrak{F} of g . The proof of the existence of such a family is like that given by Bliss and Hestenes (XVII, pp. 322-3) and by Morse (VII, p. 576) with help of Lemma 7:1. Let $p_i(x, y), \lambda_\beta(x, y)$ be the slope functions and the multipliers of the field. It is clear that the field \mathfrak{F} can be taken so small that the elements $[x, y, p(x, y), \lambda(x, y)]$ will lie in the neighborhood \mathfrak{R} specified by the condition $II\mathfrak{R}'$. The inequality (6:4) then holds at each point in \mathfrak{F} .

The identity (6:2) follows at once from the transversality condition (2:2). In order to show that the quadratic form (6:3) is positive definite on g it is convenient to express I^* in terms of the variables x, a_1, \dots, a_n instead of the variables x, y_1, \dots, y_n . In doing so we replace the functions $p_i(x, y), \lambda_\beta(x, y)$ by the functions $y_{i,x}(x, a), \lambda_\beta(x, a)$, where $\lambda_\beta(x, a)$ are the multipliers belonging to the family (9:1). We use the following abbreviations:

$$\begin{aligned} \delta y_i &= y_{ia_k} da_k, & \delta z_i &= z_{ia_k} da_k, & dy_i &= y_{i,x} dx + \delta y_i, \\ d y_i' &= y_{i,x x} dx + \delta y_i', & d^2 y_i &= y_{i,x x} d^2 x + d y_i' dx + \delta y_i' dx + \delta^2 y_i. \end{aligned}$$

With the help of the Euler-Lagrange equations (2:3) it is found that

$$d(F_{y_i'}) = F_{y_i'} dx + \delta z_i.$$

Hence at the initial point 1 on g we have

$$\begin{aligned} -dI^* + d\theta^1 &= -(F - y_i' F_{y_i'}) dx - F_{y_i'} d y_i + d\theta^1, \\ -d^2 I^* + d^2 \theta^1 &= -(F - y_i' F_{y_i'}) d^2 x - F_{y_i'} d^2 y_i - (F_x - y_i' F_{y_i'}) (dx)^2 \\ &\quad - 2F_{y_i'} d y_i dx - \delta z_i \delta y_i + d^2 \theta^1. \end{aligned}$$

If now we replace the first and second differentials of x and y_i by their values in terms of the differentials of α_μ it is found with the help of equations (3:5), (9:2), and Lemma 7:1 that the inequality

$$-d^2I^* + d^2\theta^1 = b_{\nu\nu} d\alpha_\nu d\alpha_\nu - \delta y_i \delta z_i > 0 \quad (\mu, \nu = 1, \dots, \rho)$$

holds at the point 1 on g for every set $(da_k, d\alpha_\mu) \neq (0, 0)$ satisfying the conditions $\delta y_i(x^1) = c_{i\mu} d\alpha_\mu$. By a similar argument it can be shown that at the final end point 2 on g the inequality

$$d^2I^* - d^2\theta^2 = -b_{\sigma\sigma} d\alpha_\sigma d\alpha_\sigma + \delta y_i \delta z_i > 0 \quad (\sigma, \tau = \rho + 1, \dots, r)$$

holds for every set $(da_k, d\alpha_\sigma) \neq (0, 0)$ satisfying the equations $\delta y_i(x^2) = c_{i\sigma} d\alpha_\sigma$. The last two inequalities show that the quadratic form (6:3) is positive definite on g . Theorem 6:1 now justifies the theorem that was to be proved.

Theorem 9:1 will now be used in order to obtain sufficient conditions for the problem of Bolza with end conditions of the type (1:3). In the following theorem it should be noted that the assumptions of regularity of the end conditions and of non-tangency are not needed.

THEOREM 9:2. *If an admissible arc g without corners satisfies the conditions I, II η' , III' with a set of multipliers $\lambda_0 = 1, \lambda_\rho(x)$ and if the second variation $J_2(\eta, w)$ is positive definite along g , then g affords a proper strong relative minimum to the functional J .*

In order to prove the theorem we note first that a problem of Bolza with end conditions of the form (1:3) is equivalent to the problem of finding in the class of arcs

$$y_j = y_j(x) \quad (x^1 \leq x \leq x^2; j = 1, \dots, n + r)$$

and sets α_h, γ_h ($h = 1, \dots, r$) satisfying the differential equations and end conditions

$$\begin{aligned} \phi_\beta(x, y_i, y_i') &= 0, & y_{n+h}' &= 0 & (i &= 1, \dots, n; h = 1, \dots, r), \\ x^1 &= x^1(\alpha), & y_i^1 &= y_i^1(\alpha), & y_{n+h}^1 &= \alpha_h, \\ x^2 &= x^2(\gamma), & y_i^2 &= y_i^2(\gamma), & y_{n+h}^2 &= \gamma_h \end{aligned}$$

one which minimizes the functional J . This equivalence follows from the fact that the functions $y_{n+h}(x)$ are all constants and hence take the values $\alpha_h = \gamma_h$ at $x = x^1$ and $x = x^2$. The new problem is a problem of Bolza with separated end conditions and will be called the *transformed problem*. Let g_1 be the admissible arc for the transformed problem which corresponds to the arc g of the theorem. It is easily seen that the new end conditions are regular and that the non-tangency condition holds on g_1 . The arc g_1 also satisfies the con-

ditions I, II \mathfrak{N}' , III' for the transformed problem with the set of multipliers $\lambda_0 = 1, \lambda_\beta(x), \lambda_{m+h}(x)$, where the multipliers $\lambda_{m+h}(x)$ are constants determined by the transversality condition (2:2). Moreover there is one-to-one correspondence between the admissible variations for the two problems and along corresponding admissible variations the values of the second variation for the two problems are the same. Theorems 8:1 and 9:1 therefore tell us that g_1 furnishes a proper strong relative minimum for the transformed problem and hence that g furnishes a proper strong relative minimum for the original problem, as was to be proved.

From Theorems 8:2 and 9:2 we obtain the following result:

THEOREM 9:3. *If an admissible arc g without corners satisfies the conditions I', II \mathfrak{N}' , III', V' with a set of multipliers $\lambda_0 = 1, \lambda_\beta(x)$, then g affords a proper strong relative minimum to the functional J .*

Combining Theorems 8:3 and 9:2 we obtain the further

THEOREM 9:4. *The results of Theorem 9:3 remain valid when the condition V' is replaced by the condition VI' provided that the order q of anormality of g is the same on every sub-interval x^1x^3 of the interval x^1x^2 determined by the end points of g .*

The last two theorems are extensions of the sufficient conditions given by Bliss (IX, p. 271). The following theorem gives an extension of the sufficient conditions given by Morse (VIII, p. 528) and Hu (XIX, p. 417) and is obtained by combining Theorems 8:4 and 9:2.

THEOREM 9:5. *Suppose that the end conditions are regular and that the non-tangency condition holds on an admissible arc g having no corners. If g satisfies the conditions I, II \mathfrak{N}' , III' with a set of multipliers $\lambda_0 = 1, \lambda_\beta(x)$ and if the characteristic roots of the accessory boundary value problem are all positive, then g affords a proper strong relative minimum to the functional J .*

Sufficient conditions for a weak relative minimum can be obtained in the usual manner by omitting the condition II \mathfrak{N}' in the above theorems (V, pp. 736-7).

The following example shows clearly that the sufficient conditions here given actually do not imply the normality relations which we have proposed to exclude. Let h be a small positive constant and let $A(x), B(x)$ be functions satisfying the conditions

$$\begin{aligned} A(x) > 0 \text{ on } x^1 \leq x < x^1 + h, \quad A(x) \equiv 0 \text{ on } x^1 + h \leq x \leq x^2, \\ B(x) \equiv 0 \text{ on } x^1 \leq x \leq x^2 - h, \quad B(x) > 0 \text{ on } x^2 - h < x \leq x^2 \end{aligned}$$

and having continuous derivatives of the first three orders. The segment g of the x -axis between x^1 and x^2 then furnishes a proper strong minimum to the integral

$$J = \int_{x^1}^{x^2} (1 + y_1'^2)^{1/2} dx$$

in the class of arcs (1:1) with $n=4$ satisfying the differential equations

$$y_2' = y_1' + A(x)y_1, \quad y_3' = y_1' + B(x)y_1, \quad y_4' = 0$$

and joining the two fixed points $(x, y) = (x^1, 0)$ and $(x, y) = (x^2, 0)$. The order p of anormality of g is readily found to be unity. The order q of anormality of g is unity on every sub-interval $x'x''$ satisfying the conditions $x^1 \leq x' < x^1 + h$, $x^2 - h < x'' \leq x^2$. If one of these conditions holds, then $q=2$. If neither holds, then $q=3$. It follows that the sufficient conditions given heretofore are not applicable to g . However g satisfies the sufficient conditions here given with the set of multipliers $\lambda_0 = 1$, $\lambda_\beta(x) \equiv 0$, except for those in Theorem 9:4.

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